

TRACE PALEY-WIENER THEOREM IN THE TWISTED CASE

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ABSTRACT. A version of the trace Paley-Wiener theorem for a reductive p -adic group in the context of twisted harmonic analysis with respect to an outer automorphism is proved.

Let \mathbf{G} be a connected reductive algebraic group over a p -adic field F (of characteristic 0) and let $G = \mathbf{G}(F)$. Let P_0 be a minimal parabolic subgroup of G . Let ε be an outer automorphism of \mathbf{G} of finite order r of G and suppose that ε fixes P_0 . Elements $x, y \in G$ are called ε -conjugate if $y = g^{-1}x\varepsilon(g)$ for some $g \in G$. A distribution on G which is invariant under twisted conjugacy is called ε -invariant. The objects of study in twisted harmonic analysis are ε -invariant distributions. The main examples are provided by twisted orbital integrals (invariant integrals over twisted conjugacy classes) and by twisted characters, which are obtained as follows. Let π be an irreducible admissible representation of G and let $\varepsilon(\pi)$ denote the representation of G through which $g \in G$ acts by $\pi(\varepsilon^{-1}(g))$. If π is equivalent to $\varepsilon(\pi)$, then there exists an operator $\pi(\varepsilon)$, unique up to r th roots of unity, such that $\pi(\varepsilon)\pi(g)\pi(\varepsilon)^{-1} = \pi(\varepsilon(g))$ and $\pi(\varepsilon)^r = 1$. In other words, π extends to the semidirect product $G^* = G \rtimes \langle \varepsilon \rangle$. The twisted character of π (which depends on the choice of $\pi(\varepsilon)$) is the distribution $f \rightarrow \text{Tr}(\pi(f)\pi(\varepsilon))$ on the space $\mathcal{H}(G)$ of compactly supported locally constant functions on G .

In applications of the trace formula to global problems in the theory of automorphic representations, it is often necessary to use results on the representation theory and harmonic analysis of reductive groups over p -adic fields. Because of the role played by twisted local harmonic analysis in applications of the twisted trace formula (e.g., [1, 10]), it is important to have available extensions to the twisted case of results known for ordinary harmonic analysis.

Many results in harmonic analysis extend to the twisted case. For example, by [7], Harish-Chandra's theorem on the local integrability of characters holds for twisted characters. The purpose of this paper is to prove a twisted version (Proposition 11.2) of the trace Paley-Wiener theorem for p -adic groups of Bernstein, Deligne, and Kazhdan, given in [3]. The twisted Paley-Wiener theorem gives a characterization of functions of the form $\pi \rightarrow \text{Tr}(\pi(f)\pi(\varepsilon))$ for some $f \in \mathcal{H}(G)$.

In §§1–6, the appropriate setting for the twisted theorem is developed. In §§7–10, results needed for the proof of Proposition 11.2 are proved. Proposition 7.4 is a twisted version of Casselman's theorem on characters. The proof of Proposition 11.2 follows the plan of the argument given in [3], with modifications for the twisted

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case introduced as needed. In particular, in §10, a finiteness theorem for ε -discrete representations is proved. In the ordinary case, a representation whose character cannot be written as a linear combination of characters of representations induced from proper parabolic subgroups is called discrete (or elliptic, because, by a theorem of D. Kazhdan, π is discrete if and only if its character is not identically zero on the elliptic regular set). The twisted analogue “ ε -discrete” is defined in §9. The end of a proof is denoted by a symbol. \square

1. Let $P_0 = M_0 N_0$ be a Levi decomposition of the minimal parabolic subgroup P_0 of G . We are assuming that $\varepsilon(P_0) = P_0$ and hence $\varepsilon(M_0) = n^{-1} M n$ for some $n \in N$ such that $n\varepsilon(n) \cdots \varepsilon^{r-1}(n) = 1$. The group N is an extension of additive groups and it is easily seen that $n = n_1^{-1} \varepsilon(n_1)$ for some $n_1 \in N$. Replacing M_0 by a conjugate if necessary, we may assume that $\varepsilon(M_0) = M_0$. Let A_0 be the maximal split torus in the center of M_0 .

A subgroup M of G will be called a Levi subgroup if $M \supset M_0$ and M is a Levi factor of a parabolic subgroup. A Levi factor M will be called standard if $M P_0$ is a parabolic subgroup and M is a Levi factor of $M P_0$. A parabolic subgroup of the form $M P_0$ with M standard will be called a standard parabolic subgroup. Let $\mathcal{L}(G)$ be the set of standard Levi subgroups. For $M \in \mathcal{L}(G)$, we define standard Levi subgroups of M with respect to M_0 and the minimal parabolic subgroup $M \cap P_0$ of M .

Let $N_M(A_0)$ be the normalizer of A_0 in M and let $W_M = N_M(A_0)/M_0$ be the Weyl group of M . Let A_M denote the split component of M , and let Z_M be the center of M . For H an algebraic group over F , let $X(H)$ be the group of rational characters of H . Set $\mathcal{A}(M) = \text{Hom}(X(A_M), \mathbf{Z}) \otimes \mathbf{C}$.

If M is an ε -invariant Levi subgroup, set $M^* = M \rtimes \langle \varepsilon \rangle$. The set of ε -invariant standard Levi subgroups will be denoted by $\mathcal{L}(G)^\varepsilon$. Note that if $M \in \mathcal{L}(G)^\varepsilon$, then the parabolic subgroup $M P_0$ is ε -invariant.

Let $M \in \mathcal{L}(G)$. A smooth representation (π, V) of M will be called an M -module. We will call π square-integrable if π is unitary and the matrix coefficients of π are square-integrable modulo Z_M . Similarly, we use the term tempered to mean tempered and unitary. Suppose that $M \in \mathcal{L}(G)^\varepsilon$. A representation of M^* will be called an M^* -module (resp. admissible, tempered, etc.) if its restriction to M is smooth (resp. admissible, tempered, etc.). An M^* -module is thus an M -module (π, V) together with an operator $\pi(\varepsilon)$ of order r such that $\pi(\varepsilon(m)) = \pi(\varepsilon)\pi(m)\pi(\varepsilon)^{-1}$. Let $\text{Irr}(M)$ (resp. $\text{Irr}(M^*)$) be the set of equivalence classes of irreducible M -modules (resp. M^* -modules).

2. If ζ is an r th root of unit, and (π, V) is an M^* -module, let (π_ζ, V_ζ) be the M^* -module such that $V_\zeta = V$, $\pi|_M = \pi_\zeta|_M$, and $\pi_\zeta(\varepsilon) = \zeta\pi(\varepsilon)$. Let $R_0(M^*)$ be the Grothendieck group of M^* -modules of finite length. Let $R(M^*)$ be the quotient of $R_0(M^*) \otimes \mathbf{C}$ modulo the relations

$$\pi_\zeta - \zeta\pi = 0$$

for all M^* -modules π and all r th roots of unity ζ .

If (π, V) is an M -module, let $(\varepsilon(\pi), \varepsilon(V))$ denote the module with the same underlying space V on which M acts by $m \rightarrow \pi(\varepsilon^{-1}(m))$.

LEMMA 2.1. *If $(\pi, V) \in \text{Irr}(M^*)$, then there is an irreducible M -submodule (π_0, V_0) of π and an integer t dividing r such that:*

- (a) $\pi_0, \varepsilon(\pi_0), \dots, \varepsilon^{t-1}(\pi_0)$ are inequivalent and $\varepsilon^t(\pi_0)$ is equivalent to π_0 .
- (b) $\pi|_M$ is isomorphic to $\pi_0 \oplus \varepsilon(\pi_0) \oplus \dots \oplus \varepsilon^{t-1}(\pi_0)$.
- (c) As an M^* -module, V is isomorphic to $V_0 \oplus \varepsilon(V_0) \oplus \dots \oplus \varepsilon^{t-1}(V_0)$, where ε acts by $(v_0, v_1, \dots, v_{t-1}) \rightarrow (Tv_{t-1}, v_1, \dots, v_{t-2})$ for some linear operator T on V_0 such that $T\pi_0(m)T^{-1} = \pi_0(\varepsilon^t(m))$ for $m \in M$ and $T^r/t = 1$.

PROOF. The restriction of π to M is of finite length and hence contains an irreducible M -submodule V' . Let V_0 be the sum of all irreducible submodules of V which are isomorphic to V' . Let t be the least integer such that $\varepsilon^t(V')$ is isomorphic to V' . The operator $\pi(\varepsilon)^t$ preserves V_0 . The sum $V_0 + \pi(\varepsilon)(V_0) + \dots + \pi(\varepsilon)^{t-1}(V_0)$ is direct and M^* -stable, and hence is all of V . Let T be the restriction of $\pi(\varepsilon)^t$ to V_0 . The map

$$(v_0, v_1, \dots, v_{t-1}) \rightarrow v_0 + \pi(\varepsilon)v_1 + \dots + \pi(\varepsilon)^{t-1}v_{t-1}$$

defines an M -isomorphism $V_0 \oplus \varepsilon(V_0) \oplus \dots \oplus \varepsilon^{t-1}(V_0) \rightarrow V$ which also commutes with the action of $\pi(\varepsilon)$, defined on the source as in the statement of the lemma. The irreducibility of V_0 follows from that of V . \square

Let $\text{Irr}(M^*)^0$ be the set of $\pi \in \text{Irr}(M^*)$ such that $t > 1$, where t is defined as in the previous lemma. Let $\text{Irr}(M)^\varepsilon$ be the set of $\pi \in \text{Irr}(M)$ such that $\varepsilon(\pi) = \pi$. For each $\pi \in \text{Irr}(M)^\varepsilon$, we choose, once and for all, an extension of π to M^* and regard $\text{Irr}(M)^\varepsilon$ as a subset of $\text{Irr}(M^*)$. Then $R(M^*)$ is generated by the union of $\text{Irr}(M)^\varepsilon$ and $\text{Irr}(M^*)^0$. Let $R(M^*)^0$ be the subgroup of $R(M^*)$ generated by $\text{Irr}(M^*)^0$.

The group M^*/M is cyclic of order r . If t divides r , let Σ_t be the permutation representation of M^*/M of dimension t .

LEMMA 2.2. *Let π_0 be an M^* -module of finite length and set $\pi = \pi_0 \otimes \Sigma_t$, where t divides r . Then the image of π in $R(M^*)$ lies in $R(M^*)^0$.*

PROOF. Let μ be a primitive t th root of unity. Then π is isomorphic to the direct sum

$$(\pi_0) \oplus (\pi_0)_\mu \oplus \dots \oplus (\pi_0)_{\mu^{t-1}}$$

and thus maps to zero in $R(M^*)$. \square

COROLLARY 2.3. *Let (π, V) be an M^* -module of finite length with the following property: there is an M -stable subspace V_0 of V and an integer $t > 1$ dividing r such that $V = V_0 \oplus \pi(\varepsilon)(V_0) \oplus \dots \oplus \pi(\varepsilon)^{t-1}(V_0)$ and $\pi(\varepsilon)^t V_0 = V_0$. Then the image of (π, V) in $R(M^*)$ lies in $R(M^*)^0$.*

PROOF. Let W be an irreducible M -submodule of V_0 . Let s be the least integer such that $\varepsilon^s(W)$ is equivalent to W . Let $U = W \oplus \pi(\varepsilon)(W) \oplus \dots \oplus \pi(\varepsilon)^{s-1}(W)$ and let $U_{j+1} = \pi(\varepsilon)^{js}U$. As an M -module, U is a direct sum of distinct irreducible modules and hence $U_i \cap U_j \neq \{0\}$ if and only if $\pi(\varepsilon)^{is}W = \pi(\varepsilon)^{js}W$, that is, if and only if $U_i = U_j$. We may choose k such that U_1, \dots, U_k are distinct and $U_{k+1} = U_1$. The subspace $V' = U_1 \oplus \dots \oplus U_k$ is M^* -stable. If $s > 1$, V' is a direct sum of elements of $\text{Irr}(M^*)^0$. If $s = 1$, extend U to an M^* -module π_0 such that $\pi_0(\varepsilon)^k$ is the restriction of $\pi(\varepsilon)^k$ to U . Then V' is isomorphic to $\pi_0 \otimes \Sigma_k$. The corollary follows from the lemma and induction on the length of π . \square

3. Let $L, M \in \mathcal{L}(G)$ and suppose that $L \subset M$. Let i_{ML} be the standard induction functor from L -modules to M -modules. If (ρ, V) is an L -module, then $i_{ML}(\rho)$ denotes the module unitarily induced from the parabolic subgroup $L(M \cap N_0)$ of M . The space of π consists of functions $\varphi: M \rightarrow V$ such that $\varphi(mng) = \delta_N(m)^{1/2} \rho(m) \varphi(g)$ for $m \in L$, $n \in N$, $g \in M$, where N is the unipotent radical of $L(M \cap N_0)$ and δ_N is the modulus character of L with respect to N .

If $L, M \in \mathcal{L}(G)^\varepsilon$ is a subgroup of M and if (ρ, V) is an L^* -module, then $\pi = i_{ML}(\rho|L)$ has a natural M^* -module structure. The operator $\pi(\varepsilon)$ is defined by $(\pi(\varepsilon)\varphi)(g) = \rho(\varepsilon)\varphi(\varepsilon^{-1}(g))$. This M^* -module will also be denoted by $i_{ML}(\rho)$.

Similarly, r_{LM} will denote the standard Jacquet functor. If (π, V) is an M -module, the space of $r_{LM}(\pi)$ is $V_N = V/V(N)$ where $V(N)$ is the span of vectors of the form $\pi(n)v - v$ for $n \in N$ and $v \in V$. An element $m \in L$ acts by $v + V(N) \rightarrow \delta_N(m)^{-1/2} \pi(m)v + V(N)$. If $L, M \in \mathcal{L}(G)^\varepsilon$ and if π is an M^* -module, then $V(N)$ is ε -stable since $\varepsilon(N) = N$ and thus ε acts on V_N . This makes $r_{LM}(\pi)$ into an L^* -module which will also be denoted by $r_{LM}(\pi)$.

We obtain morphisms $i_{ML}: R_0(L^*) \rightarrow R_0(M^*)$ and $r_{LM}: R_0(M^*) \rightarrow R_0(L^*)$. As in the nontwisted case, r_{LM} is left adjoint to i_{ML} . If (π, V) and (ρ, W) are M^* and L^* -modules, respectively, and $\varphi: V_N \rightarrow W$ is a map of L^* -modules, we obtain the corresponding M^* -module map $\varphi': V \rightarrow i_{ML}(W)$ by sending $v \in V$ to the function $\varphi'_v: m \rightarrow \varphi(\pi(m)v)$. Now $\varepsilon(v)$ is mapped to the function

$$\varphi'_{\varepsilon(v)}(m) = \varphi(\pi(m)\varepsilon(v)) = \varphi(\pi(\varepsilon)\pi(\varepsilon^{-1}(m))v) = \rho(\varepsilon)\varphi(\pi(\varepsilon^{-1}(m))v)$$

and this is equal to $i_{ML}(W)(\varepsilon)(\varphi'_v)(m)$. This defines a map from $\text{Hom}_{L^*}(V_N, W)$ to $\text{Hom}_{M^*}(V, i_{ML}(W))$ which is easily seen to be a bijection. The functors i_{ML} and r_{ML} induce morphisms on $R(L^*)$ and $R(M^*)$. However, i_{ML} is not necessarily injective on $R(L^*)$.

4. An element of $\text{Hom}_{\mathbb{C}}(R(M^*), \mathbb{C})$ is determined by its values on the elements of $\text{Irr}(M^*)$. Let $R(M^*)'$ be the subspace of elements of $\text{Hom}_{\mathbb{C}}(R(M^*), \mathbb{C})$ which vanish on $\text{Irr}(M^*)^0$. An element of $R(M^*)'$ is determined by its values on $\text{Irr}(M^*)^\varepsilon$.

LEMMA 4.1. *If $F \in R(M^*)'$, then the function $\rho \rightarrow F(i_{ML}(\rho))$ lies in $R(L^*)'$. If $F \in R(L^*)'$, then the function $\pi \rightarrow F(r_{LM}(\pi))$ lies in $R(M^*)'$.*

PROOF. For the first assertion, observe that if $\rho \in \text{Irr}(M^*)^0$, then $i_{ML}(\rho)$ satisfies the hypothesis of Corollary 2.3. Hence $i_{ML}(\rho) \in R(M^*)^0$ and $F(i_{ML}(\rho)) = 0$. It is clear that r_{LM} maps $\text{Irr}(M^*)^0$ to $R(L^*)^0$ and the second assertion follows. \square

This lemma shows that i_{ML} and r_{LM} define maps $i_{ML}^*: R(L^*)' \rightarrow R(M^*)'$ and $r_{LM}^*: R(M^*)' \rightarrow R(L^*)'$.

5. For $M \in \mathcal{L}(G)$, let $\mathcal{H}(M)$ be the Hecke algebra of locally constant functions on M of compact support. For K an open compact subgroup of M , let $\mathcal{H}_K(M)$ be the subalgebra of $\mathcal{H}(M)$ consisting of the bi- K -invariant elements. Every open compact subgroup contains an ε -invariant one and hence $\mathcal{H}(M)$ is the union of the $\mathcal{H}_K(M)$ for K ε -invariant.

Fix a choice of Haar measure dm on M . For $f \in \mathcal{H}(M)$ define

$$\chi_f(\pi) = \text{Trace}(\pi(f)\pi(\varepsilon)),$$

where $\pi(f) = \int f(m)\pi(m)dm$. It is clear that $\chi_f(\pi) = 0$ if $\pi \in \text{Irr}(M^*)^0$, and hence $\chi_f \in R(M^*)'$.

Let $F_{\text{tr}}(M)$ be the subspace of $R(M^*)'$ of functions of the form χ_f . Elements of $F_{\text{tr}}(M)$ will be called trace functions. As in the nontwisted case, the Paley-Wiener theorem gives a characterization of $F_{\text{tr}}(G)$ in $R(G^*)'$.

6. Let $Z(G)$ denote the Bernstein center of G [2]. Recall that an element $z \in Z(G)$ may be defined as a collection $\{z(\pi) \in \text{End}_M(\pi)\}$, where π ranges over all M -modules, such that the assignment $\pi \rightarrow z(\pi)$ commutes with all maps of G -modules. In particular, z defines an endomorphism of $\mathcal{H}(G)$. If $\pi \in \text{Irr}(G)$, then $z(\pi)$ is a scalar.

A cuspidal pair is a pair (M, ρ) where $M \in \mathcal{L}(G)$ and ρ is a supercuspidal representation of M . The set of all cuspidal pairs up to conjugation by W will be denoted by $\Theta(G)$. By definition, the image of the map $X(M) \rightarrow \Theta(G)$ defined by $\psi \rightarrow (M, \rho\psi)$ for a cuspidal pair (M, ρ) is called a connected component of $\Theta(G)$. The image of $X(M)$ in $\Theta(G)$ is given the structure of complex affine algebraic variety as a quotient of $X(M)$. According to Bernstein's theorem [2], for (M, ρ) cuspidal and $\psi \in X(M)$, an element $z \in Z(G)$ acts by a scalar on $i_{GM}(\chi\rho)$ whose dependence on χ is polynomial. This defines an isomorphism of $Z(G)$ with the algebra of regular functions on $\Theta(G)$. For $M \in \mathcal{L}(G)$, $\Theta(M)$ and $Z(M)$ are defined similarly.

If $L, M \in \mathcal{L}(G)$ and $L \subset M$, then a homomorphism $i_{ML}^*: Z(M) \rightarrow Z(L)$ is defined in [3] as the adjoint to the natural map $\Theta(L) \rightarrow \Theta(M)$. Since a cuspidal pair in $\Theta(L)$ may be conjugate under W_M but not under W_L , the map is not an inclusion. However, $Z(L)$ becomes a finitely-generated $Z(M)$ -module via i_{ML}^* .

Assume that $M \in \mathcal{L}(G)^\varepsilon$. Then the vector space $R(M^*)'$ has a natural $Z(M)$ -module structure. For $z \in Z(M)$ and $F \in R(M^*)'$, zF is the function whose value is $z(\pi)F(\pi)$ at $\pi \in \text{Irr}(M)^\varepsilon$. If $f \in \mathcal{H}(M)$ and $z \in Z(M)$, then

$$\chi_{zf}(\pi) = \text{Trace}(\pi(zf)\pi(\varepsilon)) = z(\pi)\text{Trace}(\pi(f)\pi(\varepsilon))$$

and hence $z\chi_f = \chi_{zf}$. In particular, $F_{\text{tr}}(M)$ is a $Z(M)$ -submodule of $R(M^*)'$. If $L \in \mathcal{L}(G)^\varepsilon$ and $L \subset M$, then $R(L^*)'$ has the structure of a $Z(M)$ -module via the homomorphism $Z(M) \rightarrow Z(L)$.

Let $z \in Z(M)$. For each L -module ρ , $i_{ML}^*(z)$ defines an endomorphism of ρ and induces an endomorphism $i_{ML}(i_{ML}^*(z))$ on $i_{ML}(\rho)$ by functoriality. By Proposition 2.4 of [3], $i_{ML}(i_{ML}^*(z)) = z$ on $i_{ML}(\rho)$. Similarly, for each M -module π , z induces an endomorphism $r_{LM}(z)$ on $r_{LM}(\pi)$ and Proposition 2.4 of [3] asserts that $r_{LM}(z) = i_{ML}^*(z)$ on $r_{LM}(\pi)$. This implies that the morphisms

$$i_{ML}^*: R(M^*)' \rightarrow R(L^*)', \quad r_{LM}^*: R(L^*)' \rightarrow R(M^*)'$$

are maps of $Z(M)$ -modules.

7. The following propositions is the twisted analogue of Proposition 3.2 of [3].

PROPOSITION 7.1. *For all $M \in \mathcal{L}(G)^\varepsilon$, $r_{GM}^*(F_{\text{tr}}(M)) \subset F_{\text{tr}}(G)$.*

Let $M \in \mathcal{L}(G)^\varepsilon$. Let $P = MP_0$ and let N be the unipotent radical of P . Let $P^- = MN^-$ be the opposite parabolic subgroup. An element $m \in M$ is said to be strictly contracting on N if for any two open compact subsets U_1, U_2 of N , there exists a positive integer k such that $\text{Ad}(m^k)U_1 \subset U_2$. Let M^+ be the set of elements $m \in M$ such that $\text{Ad}(m)$ is strictly contracting on N .

Let K denote a small (ε -invariant) open compact subgroup of G which is “good” in the sense that the following decomposition holds:

$$K = (K \cap N^-)(K \cap M)(K \cap N).$$

The decomposition of an element $g \in K$ as n^-mn is unique and hence the intersection of good open compact subgroups is again good. Arbitrarily small good open compact subgroups K are known to exist, and if K is good, then $\bigcap \varepsilon^i(K)$ is good and ε -invariant. Hence arbitrarily small good ε -invariant open compact subgroups K exist.

Set $K^0 = K \cap M$, $K^+ = K \cap N^-$, $K^- = K \cap N$. It will suffice to prove that $r_{MG}^*(\chi_f) \in F_{\text{tr}}(G)$ for all $f \in \mathcal{H}_{K^0}(M)$ for K ε -invariant and good.

For a set S , let $\text{ch}(S)$ denote the characteristic function of S . Let δ_N be the modulus character on M with respect to N . For $m \in M$, define functions $f'[m] \in \mathcal{H}_{K^0}(M)$ and $f[m] \in \mathcal{H}_K(G)$ by

$$\begin{aligned} f'[m] &= \text{meas}(K^0 m K^0)^{-1} \text{ch}(K^0 m K^0), \\ f[m] &= \text{meas}(K m K)^{-1} \delta_N^{-1/2}(m) \text{ch}(K m K). \end{aligned}$$

Let (π, V) be an admissible G^* -module and let $\rho = r_{MG}(\pi)$. Let $W = V_N = V/V(N)$ and let $T: V \rightarrow V_N = V/V(N)$ be the natural map. Note that T commutes with the action of ε .

LEMMA 7.2. *Let $m \in M^+$ and $w \in V^K$. Then $T(\pi(f[m])w) = \rho(f'[m])T(w)$.*

PROOF. We have

$$\text{meas}(K m K)^{-1} \int_{K m K} \pi(x)w \, dx = \text{meas}(K)^{-1} \int_K \pi(k)\pi(m)w \, dk.$$

Since $m^{-1}K^-m \subset K^-$, $\pi(m)w$ is fixed by K^- . Using the decomposition $K = K^+K^0K^-$, we obtain

$$\text{meas}(K^+)^{-1} \text{meas}(K^0)^{-1} \int_{K^+} \int_{K^0} \pi(n)\pi(k)\pi(m)w \, dn \, dk$$

and, modulo $V(N)$, this is equal to

$$\text{meas}(K^0)^{-1} \int_{K^0} \pi(k)\pi(m)w \, dk = \text{meas}(K^0 m K^0)^{-1} \int_{K^0 m K^0} \pi(x)w \, dx. \quad \square$$

Fix $a \in Z_M \cap M^+$. It is easily verified (cf. [3, §5.3]) that for all $m \in M^+$, $f[a^n m] = f[a]^n f[m]$ for $n \geq 0$ and $f'[a^n m] = f'[a]^n f'[m]$ for all n . By Proposition 3.3 of [6], for some large positive integer l , the subspace $U = \pi(f[a^l])V^K$ of V^K is stable under $\pi(f[a])$ and T maps U isomorphically to the space W^0 of K^0 -fixed vectors in W . Hence $\pi(f[a^l m])\pi(\varepsilon)V^K \subset U$ and since $T_0\pi(\varepsilon) = \rho(\varepsilon)_0 T$, we have

$$(*) \quad \text{Tr}(\pi(f[a^l m])\pi(\varepsilon) | V^K) = \text{Tr}(\pi(f[a^l m])\pi(\varepsilon) | U) = \text{Tr}(\rho(f'[a^l m])\rho(\varepsilon) | W^0)$$

by Lemma 7.2 for all $m \in M^+$.

Let \mathcal{M} be the subspace of functions f in $\mathcal{H}_K(M)$ such that $r_{MG}^*(\chi_f) \in F_{\text{tr}}(G)$. The equality $(*)$ shows that $r_{MG}^*(\chi_{f'[a^l m]}) = \chi_{f[a^l m]}$ and hence that $f'[a^l m] \in \mathcal{M}$ for all $m \in M^+$. For every $m \in M$, $a^N m \in M^+$ for some N and therefore $f'[a]^{N+l} f'[m] \in \mathcal{M}$. Since the functions $f'[m]$ span $\mathcal{H}_{K^0}(M)$, the next lemma follows.

LEMMA 7.3. *Let $a \in M^+ \cap Z_M$. Then for all $f \in \mathcal{H}_{K^0}(M)$, $f'[a]^N f \in \mathcal{M}$ for some positive integer N .*

To prove Proposition 7.1, it must be shown that $\mathcal{M} = \mathcal{H}_{K^0}(M)$. We may argue as in [3]. As shown in [2], $\mathcal{H}_{K^0}(M)$ is a finitely-generated $Z(M)$ -module, and, via the homomorphism $Z(G) \rightarrow Z(M)$, it becomes a finitely-generated $Z(G)$ -module. Since r_{LM}^* is a map of $Z(G)$ -modules and $F_{\text{tr}}(G)$ is a $Z(G)$ -submodule of $R(M^*)'$, \mathcal{M} is a $Z(G)$ -submodule of $\mathcal{H}_{K^0}(M)$. We deduce that for $a \in Z_M \cap M^+$, $f'[a]^N \mathcal{H}_{K^0}(M) = \mathcal{M}$ for some N . Since $f'[a]$ is invertible in $\mathcal{H}_{K^0}(M)$, $\mathcal{H}_{K^0}(M) = \mathcal{M}$. This completes the proof of Proposition 7.1.

Let $\pi \in \text{Irr}(G)^\varepsilon$. By [7], the distribution $f \rightarrow \text{Tr}(\pi(f)\pi(\varepsilon))$ on $\mathcal{H}(G)$ is given by integration against a locally integrable function $\chi_{\pi\varepsilon}(\gamma)$ which is locally constant on the open subset of ε -regular elements in G ($\gamma \in G$ is called ε -regular if $\gamma \rtimes \varepsilon$ is “discriminant regular” in G^* as defined in §2.2 of [7]). As was pointed out to me by L. Clozel, an analogue of Casselman’s theorem holds in the twisted case. This is given in the next proposition.

For $g \in G^*$, let P_g denote the parabolic subgroup associated to g . Recall that P_g is defined as the set of $x \in G$ such that $\{\text{ad}(g^N)x : N \in \mathbf{Z}^+\}$ is bounded in G . Observe that $P_{xgx^{-1}} = xP_gx^{-1}$ and that $g(P_g)g^{-1} = P_g$.

Fix $g = \gamma \rtimes \varepsilon \in G^*$ where $\gamma \in G$. Let $y \in G$ be such that yP_gy^{-1} is standard. Since $y^{-1}gy = y^{-1}\gamma\varepsilon(y) \rtimes \varepsilon$, we may and shall choose γ within its ε -conjugacy class so that P_g is standard. Set $P = P_g$. Then $gPg^{-1} = P$, that is, $\gamma\varepsilon(P)\gamma^{-1} = P$. But $\varepsilon(P)$ is also standard and conjugate to P , hence $\varepsilon(P) = P$ and $\gamma \in P$. Let $P = MN$ be the standard decomposition of P . If $\pi \in \text{Irr}(G)^\varepsilon$, then γ and g act on the Jacquet module π_N through the projection of γ onto M via the map $P \rightarrow P/N = M$.

PROPOSITION 7.4. *Let $(\pi, V) \in \text{Irr}(G)^\varepsilon$. Let $\gamma \in G$ be an ε -regular element and set $g = \gamma \rtimes \varepsilon$. Assume that γ is chosen within its ε -conjugacy class so that P_g is standard. Let $P = P_g = MN$ and set $\rho = r_{MG}(\pi)$. Then P is ε -invariant and $\chi_{\pi\varepsilon}(\gamma) = \delta_N(\gamma)^{1/2} \chi_{\rho\varepsilon}(\gamma)$.*

PROOF. We may imitate the proof of [6]. Let K be an ε -invariant open compact subgroup which is good with respect to $P = MN$ and is such that $\text{ad}(g)K^+ \subset K^+$, $\text{ad}(g^{-1})K^- \subset K^-$, and $\text{ad}(g)K^0 = K^0$. Let $g' = \gamma' \rtimes \varepsilon$, where γ' is the projection of γ onto M . Let $f[g]$ and $f'[g']$ be the normalized characteristic functions of KgK and $K^0g'K^0$, respectively, as before. We have that $f[g^n] = f[g]^n$ and $f'[g'^n] = f'[g']^n$ for $n \in \mathbf{Z}^+$. Let $W = V_N$. As in [6], there exists a positive integer n_0 such that for all $n \geq n_0$, the subspace $U = f[g^n]V^K$ is stable under $f[g]$ and maps injectively onto the subspace W^0 of K^0 -fixed vectors in W . The relation

$$\text{Tr}(\pi(f[g^n])|V^K) = \text{Tr}(\pi(f[g^n])|U) = \text{Tr}(\rho(f'[g'^n])|W^0)$$

holds for $n \gg 0$. Choose K small enough so that $\chi_{\pi\varepsilon}$ is constant on KmK and $\chi_{\rho\varepsilon}$ is constant on K^0mK^0 . Then $\text{Tr}(\pi(f[g])|V^K) = \delta_N(\gamma)^{-1/2} \chi_{\pi\varepsilon}(\gamma)$ and $\text{Tr}(\rho(f'[g'])|W^0) = \chi_{\rho\varepsilon}(\gamma)$. \square

8. In this section, we formulate and prove the twisted analogues of the results of §5.4 of [3].

Let $M, N \in \mathcal{L}(G)^\varepsilon$ and let $W(N, M)$ be the set of representatives for $W_N \backslash W / W_M$ of minimal length. The automorphism ε acts on W, W_M , and W_N , and preserves

the length function, since ε preserves P_0 . Hence ε acts on $W(N, M)$. Let $W(N, M)^\varepsilon$ be the set of fixed points of ε in $W(N, M)$ and let $\Omega_1, \dots, \Omega_k$ be the ε -orbits in $W(N, M)$. All elements of an orbit have the same length. Assume that the Ω_j are ordered so that the length is decreasing in j .

Let $M, N \in \mathcal{L}(G)^\varepsilon$ and let $F = r_{NG} \circ i_{MG}: R(M^*) \rightarrow R(N^*)$. For $w \in W(M, N)$, let $N(w) = N \cap wMw^{-1}$ and $M(w) = M \cap w^{-1}Nw$. If $w \in W(N, M)^\varepsilon$, then $N(w) \in \mathcal{L}(N)^\varepsilon$ and $M(w) \in \mathcal{L}(M)^\varepsilon$ and that $\rho \rightarrow i_{NN(w)} \circ w \circ r_{M(w)M}(\rho)$ is a functor from $R(M^*)$ to $R(N^*)$.

LEMMA 8.1. *For all $\rho \in R(M^*)$,*

$$F(\rho) = \sum_{w \in W(N, M)^\varepsilon} i_{NN(w)} \circ w \circ r_{M(w)M}(\rho) \quad \text{modulo } R(M^*)^0.$$

PROOF. If we forget the ε -action and regard F as a functor from the Grothendieck group of M -modules to the Grothendieck group of N -modules, then, according to [4], we have

$$F(\rho) = \sum_{w \in W(N, M)} i_{NN(w)} \circ w \circ r_{M(w)M}(\rho).$$

This is proved as follows. Let w_1, w_2, \dots, w_p be an ordering of the elements of $W(N, M)$ such that the $\text{length}(w_i) \geq \text{length}(w_{i+1})$. A functorial filtration $F_1 \subset F_2 \subset \dots \subset F_p = F$ of F is defined so that

$$F_i(\rho)/F_{i+1}(\rho) = i_{NN(w_i)} \circ w_i \circ r_{M(w_i)M}(\rho).$$

Let $P = MP_0 = MU$ and $Q = NP_0 = NU'$ (U, U' = unipotent radicals of P and Q , respectively). If V is the space of ρ , then $i_{GM}(\rho)$ acts on the space of locally constant functions $\varphi: G \rightarrow V$ such that $\varphi(mng) = \delta_U(m)^{1/2} \rho(m) \varphi(g)$. Let W_j be the subspace of those φ with support in $Pw_1Q \cup Pw_2Q \cup \dots \cup Pw_jQ$. Then W_j is Q -invariant. By definition, $F_i(\rho)$ is the image of W_j under r_{NG} .

Let $t_i = |\Omega_i|$. We may assume that w_j are ordered so that the first t_1 elements belong to Ω_1 , the next t_2 elements belong to Ω_2 , etc. Define $F'_i = F_{k(i)}$ where $k(i) = t_1 + t_2 + \dots + t_i$. Then it is clear that $F'_i(\rho)$ is invariant under ε and hence is an N^* -module. Hence $F'_i(\rho)/F'_{i-1}(\rho)$ is an N^* -module and the restriction of $F'_i(\rho)/F'_{i-1}(\rho)$ to N is a direct sum of t_i N -modules which are permuted by the action of ε . In particular, if $t_i > 1$, then $F'_i(\rho)/F'_{i-1}(\rho)$ satisfies the hypothesis of Corollary 2.3 and hence belongs to $R(N^*)^0$. \square

For $M \in \mathcal{L}(G)^\varepsilon$, let $T_M = i_{GM} \circ r_{MG}: R(G^*) \rightarrow R(G^*)$.

LEMMA 8.2. *Assume that $N, M \in \mathcal{L}(G)^\varepsilon$. Let $\rho \in R(M^*)$. Then*

$$(a) \quad T_N \circ i_{GM}(\rho) = \sum_{w \in W(M, N)^\varepsilon} i_{GM(w)} \circ r_{M(w)M}(\rho) \quad \text{modulo } R(G^*)^0.$$

$$(b) \quad T_N \circ T_M(\pi) = \sum_{w \in W(M, N)^\varepsilon} T_{M(w)}(\pi) \quad \text{modulo } R(G^*)^0.$$

PROOF. If $w \in W(M, N)^\varepsilon$, then $N(w) = wM(w)w^{-1}$. As follows from Lemma 5.4(iii) of [3], $i_{GN(w)} \circ w(\sigma) = i_{GM(w)}(\sigma)$ for all $\sigma \in R(M(w)^*)$. Part (a) follows

from Lemma 8.1. For part (b), we have

$$\begin{aligned} r_{NG} \circ i_{GM} \circ r_{MG}(\pi) &= \sum_{w \in W(M, N)^\varepsilon} i_{NN(w)} \circ w \circ r_{M(w)M}(r_{MG}(\pi)) \quad \text{modulo } R(M^*)^0 \\ &= \sum_{w \in W(M, N)^\varepsilon} i_{NN(w)} \circ w \circ r_{M(w)G}(\pi) \quad \text{modulo } R(M^*)^0 \end{aligned}$$

and hence

$$T_N \circ T_M(\pi) = \sum_{w \in W(M, N)^\varepsilon} i_{GN(w)} \circ w \circ r_{M(w)G}(\pi) \quad \text{modulo } R(M^*)^0$$

and (b) follows. \square

9. For $M \in \mathcal{L}(G)$, let $X(M)$ denote the group of unramified characters of M . Put $d(M) = \dim(X(M))$. For $m \in \mathcal{L}(G)^\varepsilon$, let $X(M)^\varepsilon$ be the set of fixed points of ε in $X(M)$. Define a decreasing filtration $\{R^i\}$ of $R(G^*)$ by

$$R^i = R(G^*)^0 + \sum_{\substack{M \in \mathcal{L}(G)^\varepsilon \\ d(M) \geq i}} i_{GM}(R(M^*)).$$

Let $R(G^*)^I = R^{d(G)+1}$. An element $\pi \in \text{Irr}(M)^\varepsilon$ will be called ε -discrete if it does not lie in $R(G^*)^I$.

Lemma 8.2(a) shows that T_N preserves the filtration $\{R^i\}$ of $R(G^*)$ for $N \in \mathcal{L}(G)^\varepsilon$. Let $P(N) = \text{Card}(\{w \in W(N, N)^\varepsilon : wNw^{-1} = N\})$. Then for $d = d(N)$, $M \in \mathcal{L}(G)^\varepsilon$, and $\sigma \in R(M^*)$:

$$T_N(i_{GM}(\sigma)) = \begin{cases} P(N)i_{GM}(\sigma) & \text{modulo } R^{d+1} \text{ if } M \text{ is conjugate to } N, \\ 0 & \text{modulo } R^{d+1} \text{ if } M \text{ is not conjugate to } N. \end{cases}$$

Set $A_d = \prod (T_N - P(N))$, the product over $N \in \mathcal{L}(G)^\varepsilon$ such that $d(N) = d$. Then A_d preserves the filtration $\{R^i\}$ and maps R^d to R^{d+1} . Hence the operator

$$A = A_{d(M_0)} \circ A_{d(M_0)} \circ \cdots \circ A_{d(G)+1}$$

maps R^I to $R(G^*)^0$. By Lemma 8.2(b), there exist $C_m \in \mathbf{Q}$ and $P \in \mathbf{Z}$ such that

$$A(\pi) = P \left(1 - \sum C_M T_M \right) (\pi) \quad \text{modulo } R(G^*)^0,$$

where the sum is over $M \in \mathcal{L}(G)^\varepsilon$.

If $F \in R(G^*)'$, we will call F ε -discrete if it vanishes on $R(G^*)^I$. Let $R(G^*)'_{\text{disc}}$ denote the subspace of ε -discrete functions in $R(G^*)'$.

PROPOSITION 9.1. *Let $F \in R(G^*)'$ and define*

$$F^d = F - \sum C_M r_{MG}^* i_{GM}^*(F).$$

Then $F^d \in R(G^)'_{\text{disc}}$.*

PROOF. By definition, $F^d(\pi) = P^{-1}F(A(\pi))$ for $\pi \in R(G^*)$. If $\pi \in R(G^*)^I$, then $A(\pi) \in R(G^*)^0$ and so $F(A(\pi)) = 0$. \square

10. Define a map $\inf: \text{Irr}(G)^\varepsilon \rightarrow \Theta(G)$ by sending $\pi \in \text{Irr}(G)^\varepsilon$ to a cuspidal pair (M, ρ) such that π is a constituent of $i_{GM}(\rho)$. The pair (M, ρ) is well defined modulo conjugation by W by the results of [4].

If $\inf(\pi) = \theta$ for some ε -discrete $\pi \in \text{Irr}(G)^\varepsilon$, then θ will be called discrete. For Θ a component of $\Theta(G)$, let Θ_{disc} be the set of discrete θ in Θ . For $M \in \mathcal{L}(G)^\varepsilon$, let $X(M)^\varepsilon$ be the subgroup of $X(M)$ fixed by ε .

The purpose of this section is to prove the following proposition.

PROPOSITION 10.1. Θ_{disc} is the union of finitely many $X(G)^\varepsilon$ -orbits.

We first prove some preliminary results.

LEMMA 10.2. *Let $\pi \in \text{Irr}(G)^\varepsilon$ be an ε -discrete representation. Then there exists an ε -discrete tempered representation $\pi' \in R(\theta)$ such that $\inf(\pi')$ is in the $X(G)^\varepsilon$ -orbit of $\inf(\pi)$.*

PROOF. By the Langlands classification, to every $\pi \in \text{Irr}(G)$ is associated Langlands data (P, σ, χ) consisting of a standard parabolic subgroup P , a tempered representation σ of the Levi factor M of P , and a character χ of M which is positive with respect to the unipotent radical of P , such that π is the unique irreducible quotient of $i_{G,M}(\chi\sigma)$. The triple (P, σ, χ) is unique. The Langlands data associated to $\varepsilon(\pi)$ is $(\varepsilon(P), \varepsilon(\sigma), \varepsilon(\chi))$. In particular, if $\pi \in \text{Irr}(G)^\varepsilon$, then (P, σ, χ) is ε -invariant. Let ρ be a supercuspidal representation of a Levi subgroup M' contained in M such that $\chi\sigma$ is a constituent of $i_{MM'}(\rho)$. Then each constituent of $i_{GM}(\chi\sigma)$ is also a constituent of $i_{GM'}(\rho)$. We have

$$\pi = i_{GM}(\chi\sigma) - \sum \pi_i \quad (\text{modulo } R(G^*)^0),$$

where the π_i are the irreducible ε -invariant constituents of $i_{GM}(\chi\sigma)$ distinct from π (provided that the choice of the actions of ε on π and the π_i are chosen compatible with that on $i_{GM}(\chi\sigma)$; we may assume this without loss of generality). Similarly, if (P_i, σ_i, χ_i) is $(\varepsilon$ -invariant) Langlands data for π_i , then $\pi_i = i_{G_i M_i}(\chi_i \sigma_i) - \sum \pi_{ji}$ (modulo $R(G^*)^0$). The process can be continued until we obtain an equality of the form

$$\pi = \sum a_j i_{G_j M_j}(\chi_j \sigma_j) + \sum b_k \pi_k \quad (\text{modulo } R(G^*)^0),$$

where $a_j, b_k \in \mathbb{Z}$, $M_j \neq G$, and the π_k are essentially tempered irreducible, ε -invariant representations (the process stops after finitely many steps by Lemma 2.13, p. 334 of [5]). The sum over π_k is nonempty since π is ε -discrete and there is at least one element π_m in the sum which is ε -discrete. The π_k are constituents of $i_{GM'}(\rho)$ (Proposition 4.7 of [11]) and hence $\inf(\pi_m) = \inf(\pi)$. Let ω be the central character of π_k . Then the character $|\omega(z)|$ is ε -invariant and there exists an element $\chi \in X(G)^\varepsilon$ whose restriction to the center Z of G is $|\omega(z)|$. Hence $\pi_m \otimes \chi^{-1}$ is an ε -discrete tempered representation of G and $\inf(\pi_m \otimes \chi) = \inf(\pi)$. \square

For $M \in \mathcal{L}(G)$, let $X(M)_{\text{u}}^\varepsilon$ be the subgroup of unitary characters in $X(M)^\varepsilon$.

PROPOSITION 10.3. *Let $M \in \mathcal{L}(G)$ and suppose that $s\varepsilon(M) = M$, where $s \in W$. Then there is a finite set V of square-integrable representations δ' of M such that $s\varepsilon(\delta') = \delta'$, and a finite union F of $X(G)_{\text{u}}^\varepsilon$ -orbits of unitary characters of M with the following property: if δ is square-integrable representation of M such*

that $s\varepsilon(\delta) = \delta$ and $i_{GM}(\delta)$ contains an ε -discrete, ε -invariant representation π such that $\inf(\pi) \in \Theta$, then $\delta = \delta' \otimes \chi$ where $\delta' \in V$ and $\chi \in F$.

PROOF. We begin with some lemmas. The first is taken from [9].

LEMMA 10.4. *Let $M \in \mathcal{L}(G)$ and let $s \in W$ such that $s\varepsilon(M) = M$. Then there exists a Levi subgroup $L \in \mathcal{L}(G)^\varepsilon$ and an element $w \in W$ such that $L \supset w^{-1}Mw$, $s\varepsilon(w) \in L$, and $w^{-1}(\mathcal{A}(M)^{s\varepsilon}) = \mathcal{A}(L)^\varepsilon$.*

PROOF. We use induction on the semisimple rank of G . If $\mathcal{A}(M)^{s\varepsilon} = \mathcal{A}(G)^\varepsilon$, the assertion is clear. Otherwise, there is an element $\chi \in \mathcal{A}(M)^{s\varepsilon}$ on which all roots do not vanish. This defines a proper parabolic subgroup P_1 containing M which is fixed by $s\varepsilon$. Choose $w \in W$ so that $P'' = w^{-1}P_1w$ is a standard parabolic subgroup and set $M' = w^{-1}Mw$. Let $s' = w^{-1}s\varepsilon(w)$. Then $s'\varepsilon(M')s'^{-1} = M'$ and $\varepsilon(P'') = s'^{-1}P''s'$. Since P'' is standard, this implies that $\varepsilon(P'') = P''$ and that $s' \in W_{M''}$, where M'' is the Levi factor of P'' . The induction assumption can be applied to M'' with M' in place of M . \square

Let $M, L \in \mathcal{L}(G)$ and suppose that $M \subset L$. Let ρ be an M -module. Since $i_{GM} = i_{GL} \circ i_{LM}$ and i_{GL} is right adjoint to r_{LG} , there is a canonical map $r_{LG} \circ i_{GM}(\rho) \rightarrow i_{LM}(\rho)$ which corresponds to the identity in $\text{Hom}(i_{GM}(\rho), i_{GM}(\rho))$. Explicitly, the map is obtained as follows. The representation $i_{GM}(\rho)$ (resp. $i_{LM}(\rho)$) acts on a space of functions on G (resp. L). Regarding $i_{GM}(\rho)$ as an L -module, there is a surjective map of L -modules $i_{GM}(\rho) \rightarrow i_{LM}(\rho)$ given by restricting a function f on G to L . This map factors to give the canonical map $r_{LG} \circ i_{GM}(\rho) \rightarrow i_{LM}(\rho)$ which is also surjective. From this we also obtain a canonical map $c: r_{MG} \circ i_{GM}(\rho) \rightarrow r_{ML} \circ i_{LM}(\rho)$.

Let τ be another M -module and let $T': i_{LM}(\rho) \rightarrow i_{LM}(\tau)$. By adjunction, we have a map $r_{LG} \circ i_{GL}(T'): r_{LG} \circ i_{GL} \circ i_{LM}(\rho) \rightarrow i_{LM}(\tau)$ from which it follows that the map $r_{MG} \circ i_{GL}(T'): r_{MG} \circ i_{GM}(\rho) \rightarrow \tau$ factors as

$$r_{MG} \circ i_{GM}(\rho) \xrightarrow{c} r_{ML} \circ i_{LM}(\rho) \xrightarrow{r_{ML}(T')} \tau.$$

In particular, a map $T: i_{GM}(\rho) \rightarrow i_{GM}(\tau)$ is induced from a map $T': i_{LM}(\rho) \rightarrow i_{LM}(\tau)$ if and only if $r_{MG}(T)$ factors through c .

Let w_1, \dots, w_t be an ordering of the set of representatives of $W_M \backslash W / W_M$ of minimal length such that $\text{length}(w_i) \geq \text{length}(w_{i+1})$. Let P be the standard parabolic subgroup with Levi factor M and let $W_j(\rho)$ be the subspace of $i_{GM}(\rho)$ consisting of functions supported in $Pw_1P \cup Pw_2P \cup \dots \cup Pw_jP$. Let $F_j(\rho)$ be the image of $W_j(\rho)$ in $r_{MG} \circ i_{GM}(\rho)$. As in §8, $F_j(\rho)/F_{j-1}(\rho)$ is isomorphic to $i_{MM(w_j)} \circ w_j \circ r_{M(w_j^{-1})M}(\rho)$ where we set $M(w) = wMw^{-1} \cap M$. Observe that if the double coset $W_M w_j W_M$ is contained in W_L , then $w_j \in W_L$.

LEMMA 10.5. *Let ρ, τ be irreducible representations of M and let $T: i_{GM}(\rho) \rightarrow i_{GM}(\tau)$ be a map of G -modules. Let $r_{MG}(T)$ be the associated map from $r_{MG} \circ i_{MG}(\rho)$ to τ . Suppose that for some j , $F_{j-1}(\rho) \in \text{Ker}(r_{MG}(T))$ and that $r_{MG}(T)$ induces a surjective map $F_j(\rho)/F_{j-1}(\rho) \rightarrow \tau$. Suppose that $w_j \in W_L$. Then $T = i_{GL}(T')$ where $T': i_{LM}(\rho) \rightarrow i_{LM}(\tau)$.*

PROOF. By the above discussion, we must show that $r_{MG}(T)$ factors through the canonical map $c: r_{MG} \circ i_{GM}(\rho) \rightarrow r_{ML} \circ i_{LM}(\rho)$. Let P' be the standard parabolic

subgroup of L whose Levi factor is M . If $w_j \in W_L$, let $Z_j(\rho)$ be the subspace of $i_{LM}(\rho)$ consisting of functions supported in the union of the double cosets $P'w_iP'$, where i ranges over the positive integers such that $i \leq j$ and $w_i \in W_L$ and let $G_j(\rho)$ be the image of $Z_j(\rho)$ in $r_{ML} \circ i_{LM}(\rho)$. If j' is the largest integer such that $j' < j$ and $w_j \in W_L$, then $G_j(\rho)/G_{j'}(\rho)$ is isomorphic to $i_{MM(w_j)} \circ w_j \circ r_{M(w_j^{-1})M}(\rho)$.

Let N' be the unipotent radical of the standard parabolic subgroup of G whose unipotent radical is L . Then $P = P'N'$ and hence if $w \in W_L$, $PwP = P'wP$. It follows that $PwP \cap L' = P'wP'$. On the other hand, if $w \in W - W_L$, then $PwP \cap L$ is empty since the Bruhat decompositions are disjoint. It follows that the L -module map $i_{GM}(\rho) \rightarrow i_{LM}(\rho)$ sends $W_j(\rho)$ onto $Z_j(\rho)$ and that c maps $F_j(\rho)$ onto $G_j(\rho)$, inducing an isomorphism of $F_j(\rho)/F_{j-1}(\rho)$ onto $G_j(\rho)/G_{j'}(\rho)$.

Let $K = \text{Ker}(r_{MG}(T))$. Then $r_{MG} \circ i_{GM}(\rho) = K + F_j(\rho)$ and $r_{ML} \circ i_{LM}(\rho) = c(K) + G_j(\rho)$ since c is surjective. If $x = k + g$, where $k \in c(K)$ and $g \in G_j(\rho)$, define $S(x) = T(f)$ where $f \in F_j(\rho)$ is any element such that $c(f) = g$. Then $S(x)$ is well defined since $T(F_{j-1}(\rho)) = 0$ and $r_{MG}(T) = S \circ c$. \square

We now prove Proposition 10.3, basing our argument on that given in the proof of Lemma 4 of [8]. For the convenience of the reader, we reproduce those parts of [loc. cit.] needed.

LEMMA 10.6. *There are only finitely-many $X(M)_u$ -orbits of square-integrable δ such that $i_{GM}(\delta)$ has a constituent π with $\inf(\pi) \in \Theta$.*

PROOF. There exists an open compact subgroup K of G such that the space π^K of K -invariants is nonzero for all $\pi \in \text{Irr}(G)$ such that $\inf(\pi) \in \Theta$ [2]. If δ is a unitary square-integrable representation of M such that $i_{GM}(\delta)$ has a constituent π with $\inf(\pi) \in \Theta$, then $\delta^{K \cap M} \neq \{0\}$ by Frobenius reciprocity. The lemma follows if we show that the set of unitary square-integrable δ such that $\delta^{K \cap M} \neq \{0\}$ is finite modulo $X(M)_u$. Let Z_M be the center of M . Since the restriction map from $X(M)_u$ to $\text{Hom}_u(Z_M, \mathbf{C}^*)$ is finite, it will suffice to show that there are only finitely many such δ with a fixed central character χ . By a theorem of Harish-Chandra (cf. [8]), the space ${}^0\mathcal{C}_\chi(M, K \cap M)$ of bi- $(K \cap M)$ -invariant functions in the space of cuspidal Schwartz functions on M which transform by χ under the Z_M is finite-dimensional. Furthermore, ${}^0\mathcal{C}_\chi(M, K \cap M)$ contains the bi- $(K \cap M)$ -invariant matrix coefficients of the square-integrable δ which transform under Z_M by χ . The finiteness follows. \square

Let δ be a unitary square-integrable representation of M such that $s\varepsilon(\delta) = \delta$ and $i_{GM}(\delta)$ has an ε -discrete, ε -invariant constituent π such that $\inf(\pi) \in \Theta$. Let $W(G, A_M) = N_G(A_M)/M$, where $N_G(A_M)$ is the normalizer of A_M in G and set $W_\delta = \{w \in W(G, A_M) : \delta = w(\delta)\}$. Since there are only finitely many possibilities for W_δ , it will suffice to prove the finiteness of the number of $X(G)_u^\varepsilon$ -orbits in the set $F(\delta)$ of $\chi \in X(M)_u$ such that $s\varepsilon(\chi\delta) = \chi\delta$, $i_{GM}(\chi\delta)$ has an ε -discrete, ε -invariant constituent π such that $\inf(\pi) \in \Theta$ and $W_{\delta\chi} = W_\delta$.

Let $\{\alpha_1, \dots, \alpha_k\}$ be a basis for the rational co-characters of A_M and let ω be a prime element in F . Let C_M be the free abelian group generated by $\{\alpha_j(\omega)\}$. Then the center of M/C_M is compact. If τ is an irreducible representation of M , let ψ_τ denote the character of C_M by which C_M acts on τ . Let $H: M \rightarrow \mathcal{A}(M)_\mathbf{R}$ denote the standard map, where $\mathcal{A}(M)_\mathbf{R} = \text{Hom}(X(A_M), \mathbf{Z}) \otimes \mathbf{R}$. The image of C_M under H is a lattice \mathcal{L} in $\mathcal{A}(M)_\mathbf{R}$. We identify $\text{Hom}(C_M, \mathbf{C}^*)$ with $\mathcal{A}(M)^*/\mathcal{L}^\wedge$,

where $\mathcal{A}(M)^* = X(A_M) \otimes \mathbb{C}$ and $\mathcal{L}^\wedge = \{\lambda \in \mathcal{A}(M)^*: \lambda(\mathcal{L}) \in (2\pi i / \log(q))\mathbb{Z}\}$, by associating to $\lambda \in \mathcal{A}(M)^*$ the character $c \rightarrow q^{(H(c), \lambda)}$. Here q is the cardinality of the residue field of F .

Suppose that $F(\delta)$ is infinite modulo the action of $X(G)_u^\varepsilon$. There are only finitely many characters $\chi \in X(M)_u$ whose restriction of C_M is trivial and hence the set $\{\psi_{\chi\delta}: \chi \in F(\delta)\}$ is infinite modulo $X(G)_u^\varepsilon$. Observe that W_δ is stable under the action of $s\varepsilon$ since $s\varepsilon(w)s^{-1}(\delta) = \delta$ if $w \in W_\delta$. Let W' be the semidirect product of W_δ with the cyclic group generated by $s\varepsilon$. Let $\text{Hom}_u(C_M, \mathbb{C}^*)$ be the group of unitary characters of C_M . Then W' acts on $\text{Hom}_u(C_M, \mathbb{C}^*)$ and $\psi_{\chi\delta}$ is fixed by W' for all $\chi \in F(\delta)$. It follows that there are infinitely many W' -fixed points in the torus $\mathcal{A}(M)^* / \mathcal{L}^\wedge$ which are not in the image of $\mathcal{A}(G)^*$. Since the action of W' on $\mathcal{A}(M)^* / \mathcal{L}^\wedge$ is algebraic, there exists a nonzero vector $\lambda \in \mathcal{A}(M)^* - \mathcal{A}(G)^*$ fixed by W' . The action of W' on $\mathcal{A}(M)^* = X(A_M) \otimes \mathbb{C}$ comes from an action of W' on $X(M)$ and hence W' fixes a sublattice V of $X(M)$ which is not contained in $X(G)$. It follows that W' fixes a subtorus A of A_M which strictly contains A_G . According to [6, Lemma 1.1], the centralizer of A is a Levi subgroup L' of G which contains M . By construction, $W(L', A_M)$ contains W_δ and L' is stable under $s\varepsilon$. By Lemma 10.4, we may conjugate the entire situation and assume that there exists a proper ε -stable Levi factor L which contains L' such that $s \in W_L$.

According to the Corollary to Theorem 2 of [8], for $\chi \in F(\delta)$, every irreducible constituent of $i_{GM}(\delta\chi)$ is of the form $i_{GL}(\rho)$, where ρ is an irreducible representation of $i_{LM}(\delta\chi)$. By hypothesis, $i_{GM}(\delta\chi)$ contains at least one ε -invariant, ε -discrete subrepresentation $\pi(\chi)$. Let $\rho(\chi)$ be the subrepresentation of $i_{LM}(\delta\chi)$ such that $i_{GL}(\rho(\chi)) = \pi(\chi)$.

Since L is ε -invariant and $s \in W_L$, the data $(M, \delta\chi)$ is conjugate to $(\varepsilon(M), \varepsilon(\delta\chi))$ in L . The set of subrepresentations of $i_{LM}(\delta\chi)$ is therefore stable under ε and so $\varepsilon(\rho(\chi))$ is also a subrepresentation of $i_{LM}(\delta\chi)$. Since $\pi(\chi)$ is ε -discrete, $\rho(\chi)$ cannot be ε -invariant, but since $\pi(\chi)$ is ε -invariant, $i_{GL}(\rho(\chi))$ is isomorphic to $i_{GL}(\varepsilon(\rho(\chi)))$. Identify $\varepsilon(\rho(\chi))$ with a subrepresentation of $i_{LM}(\delta\chi)$ and let $\varphi(\chi): i_{GM}(\delta\chi) \rightarrow i_{GM}(\delta\chi)$ be any map which sends the summand $i_{GL}(\rho(\chi)) = \pi(\chi)$ to the summand $i_{GL}(\varepsilon(\rho(\chi)))$. Then $\varphi(\chi)$ is not the identity map, it corresponds, by adjunction, to a map $\xi(\chi): r_{GM} \circ i_{GM}(\delta\chi) \rightarrow \delta\chi$ which is distinct from the standard map which corresponds by adjunction to the identity.

Let $F = \{\chi\}$ be a set of representatives for the $X(G)_u^\varepsilon$ -orbits in $F(\delta)$. Since F is infinite, there exists a nontrivial element $w \in W$ giving a shortest representative for a double coset in $W_M \backslash W / W_M$ such that $\xi(\chi)$ induces a map $i_{MM(w)} \circ w \circ r_{M(w^{-1})M}(\delta\chi) \rightarrow \delta\chi$ (where $i_{MM(w)} \circ w \circ r_{M(w^{-1})M}(\delta\chi)$ is regarded as a subquotient of $r_{GM} \circ i_{GM}(\delta\chi)$) for infinitely many $\chi \in F$. The representation $i_{MM(w)} \circ w \circ r_{M(w^{-1})M}(\delta\chi)$ is isomorphic to $w(\chi)i_{MM(w)} \circ w \circ r_{M(w^{-1})M}(\delta\chi)$ and hence $\xi(\chi)$ induces a map from $\chi^{-1}w(\chi)i_{MM(w)} \circ w \circ r_{M(w^{-1})M}(\delta)$ to δ . It follows that there are infinitely many $\chi \in F$ such that $\psi_{\chi\delta}$ is fixed by w and by W' . Let W'' be the group of automorphisms of $\text{Hom}_u(C_M, \mathbb{C}^*)$ generated by w and W' . We may repeat the above argument to obtain a proper ε -invariant Levi subgroup L' containing M such that $W_\delta \in W(L', A_M)$ and $s, w \in W_{M'}$. By Lemma 10.5, the map $\varphi(\chi)$ is obtained by induction from a map $\varphi_0(\chi): i_{L'M}(\delta\chi) \rightarrow i_{L'M}(\delta\chi)$. There are irreducible subrepresentations $\tau(\chi), \tau'(\chi)$ of $i_{L'M}(\delta\chi)$ such that $\varepsilon(\tau(\chi))$ is isomorphic to $\tau'(\chi)$, $i_{GL'}(\tau(\chi)) = i_{GL}(\rho(\chi))$, and $i_{GL'}(\tau'(\chi)) = i_{GL}(\varepsilon(\rho(\chi)))$. The

map $\varphi_0(\chi)$ must take $\tau(\chi)$ to $\tau'(\chi)$ and hence $\tau(\chi)$ is ε -invariant. This contradicts the assumption that $\pi(\chi)$ is ε -discrete and proves the proposition.

We now prove Proposition 10.1. Let $\theta \in \Theta_{\text{disc}}$. By Lemma 10.2, we may assume, modulo the action of $X(G)^\varepsilon$, that $\inf(\pi) = \theta$ for some ε -discrete tempered representation π . There exists a standard parabolic subgroup $P = MN$ and a unitary square-integrable representation δ of M such that π is a subrepresentation of $i_{GM}(\delta)$. The data (M, δ) is determined up to conjugacy by π and since $\varepsilon(\pi)$ is a subrepresentation of $i_{G\varepsilon(M)}(\varepsilon(\delta))$, there exists an element $s \in W$ such that $s\varepsilon(M) = M$ and $s\varepsilon(\delta) = \delta$. By Proposition 10.3, there are only finitely many pairs (M, δ) such that $i_{GM}(\delta)$ contains an ε -invariant, ε -discrete subrepresentation, modulo the action of $X(G)^\varepsilon$. Proposition 10.1 follows.

11. Let ε act on $\Theta(G)$ by sending (M, ρ) to $(\varepsilon(M), \varepsilon(\rho))$. Then ε permutes the set of components of $\Theta(G)$. Let $\pi \in \text{Irr}(G)^\varepsilon$ and suppose that π is a constituent of $i_{GM}(\rho)$ where (M, ρ) is a cuspidal pair. Then π is a constituent of both $i_{GM}(\rho)$ and $\varepsilon(i_{GM}(\rho)) = i_{G\varepsilon(M)}(\varepsilon(\rho))$. It follows that $\varepsilon(\inf(\pi)) = \inf(\pi)$, that is, there exists an element $s \in W$ such that $M = s\varepsilon(M)s^{-1}$ and $s\varepsilon(\rho) = \rho$. If Θ is a component of $\Theta(G)$ such that $\inf(\pi) \in \Theta$ for some $\pi \in \text{Irr}(G)^\varepsilon$, then $\varepsilon(\Theta) = \Theta$.

For Θ such that $\varepsilon(\Theta) = \Theta$, let $R(\Theta)$ be the subspace of $R(G^*)$ generated by $\{\pi \in \text{Irr}(G)^\varepsilon : \inf(\pi) \in \Theta\}$. In order to state the twisted Paley-Wiener theorem (Theorem 11.2 below), define the subspace $F_g(G)$ of elements $F \in R(G^*)'$ such that:

(I) There is a finite set S of ε -stable components of $\Theta(G)$ such that F is supported in $\sum_{\Theta \in S} R(\Theta)$.

(II) For every $L \in \mathcal{L}(G)^\varepsilon$ and $\rho \in \text{Irr}(L)^\varepsilon$, the function $\psi \rightarrow F(i_{GL}(\psi\rho))$ is a regular function on $X(L)^\varepsilon$.

For $M \in \mathcal{L}(G)^\varepsilon$, $F_g(M)$ is defined similarly.

LEMMA 11.1. (a) $F_{\text{tr}}(G) \subset F_g(G)$.

(b) For all $M \in \mathcal{L}(G)^\varepsilon$, $i_{GM}^*(F_g(G)) \subset F_g(M)$.

PROOF. Let K be an open compact subgroup of G . By [2], there are only finitely-many components Θ of $\Theta(G)$ such that $\pi^k \neq \{0\}$ for some $\pi \in \text{Irr}(G)$ such that $\inf(\pi) \in \Theta$. Hence χ_f satisfies (I) for all $f \in \mathcal{H}_K(G)$. Let $L \in \mathcal{L}(G)^\varepsilon$ and $\rho \in \text{Irr}(L)^\varepsilon$. The representations $i_{GL}(\psi\rho)$ for $\chi \in X(L)^\varepsilon$ all act on a fixed vector space V . Both the action of ε and the subspace V^K of K -fixed vectors in V is independent of χ . Furthermore, if we choose a basis for V^K , then f acts by a matrix whose coefficients are polynomials on $X(L)^\varepsilon$, and hence χ_f satisfies (II). This proves (a). To prove (b), observe that if $F \in F_g(G)$, then $i_{GM}^*(F)$ satisfies (II) by transitivity of induction and $i_{GM}^*(F)$ satisfies (I) because the canonical map from $\Theta(M)$ to $\Theta(G)$ is finite. \square

PROPOSITION 11.2. $F_{\text{tr}}(G) = F_g(G)$.

PROOF. We follow [3]. Let $F \in F_g(G)$. For Θ an ε -stable component, let $1(\Theta) \in \mathcal{Z}(G)$ be the characteristic function of Θ . Then $F = \sum 1(\Theta)F$, where all but finitely many terms in the sum are zero, and $1(\Theta)F \in F_g(G)$. To prove that $F \in F_{\text{tr}}(G)$, we may therefore assume that F is supported in a single ε -stable component Θ . We may further assume by induction that the theorem holds for all proper standard Levi subgroups of G .

Let $F_{\text{disc}}(G)$ be the space of discrete functions in $F_g(G)$. Let $R(\Theta_{\text{disc}})$ be the subspace of $R(\Theta)$ generated by the set of $\pi \in \text{Irr}(G^*)$ such that $\inf(\pi) \in \Theta$ and $\inf(\pi) = \inf(\pi')$ for some ε -discrete $\pi' \in \text{Irr}(G)^\varepsilon$. Proposition 10.1 implies that $R(\Theta_{\text{disc}})$ is generated by finitely many $X(G)^\varepsilon$ -orbits of G^* -modules and hence for all $F \in F_{\text{disc}}(G)$, there exists $F' \in F_{\text{tr}}(G)$ such that $F - F'$ vanishes on $R(\Theta_{\text{disc}})$. (If $X(G)^\varepsilon = \{1\}$, the implication follows from the linear independence of twisted characters on $\mathcal{H}(G)$; in the general case we may argue as in §4 of [3]).

Suppose that $F \in F_g(G)$ and that F vanishes on $R(\Theta_{\text{disc}})$. By Proposition 7.1 and the induction hypothesis, $r_{MG}^*(i_{GM}^*(F_g(G))) \subset F_{\text{tr}}(G)$ for $M \in \mathcal{L}(G)^\varepsilon$. Proposition 9.1 then implies that $F - F^d \in F_{\text{tr}}(G)$, where F^d is as defined in Proposition 9.1. However, $F^d = 0$ if F vanishes on $R(\Theta_{\text{disc}})$ because the operators $i_{GM} \circ r_{MG}$ preserve $R(\Theta_{\text{disc}})$.

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